

RIEMANN SURFACES IN STEIN MANIFOLDS WITH DENSITY PROPERTY

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ABSTRACT. It is shown that any open Riemann surface can be immersed in any Stein manifold with (volume) density property and of dimension at least 2, if the manifold possesses an exhaustion with holomorphically convex compacts such that their complement is connected. The immersion can be made into an embedding if the dimension is at least 3. As an application, it is shown that Stein manifolds with (volume) density property and of dimension at least 3, are characterized among all other complex manifolds by their semigroup of holomorphic endomorphisms.

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1. INTRODUCTION

An open Riemann surface always admits a proper holomorphic embedding into \mathbb{C}^3 and an proper holomorphic immersion into \mathbb{C}^2 . In this paper we generalize these results to embeddings and immersions of open Riemann surfaces into Stein manifolds with the density or volume density property. Our main result is the following:

Theorem (5.1). *Let X be a Stein manifold with density property or with volume density property and \mathcal{R} an open Riemann surface. If $\mathcal{R} \not\cong \mathbb{C}$, then assume further there exists a strongly plurisubharmonic exhaustion function τ of X with increasing compact sublevel sets $K_j = \tau^{-1}([-\infty, M_j])$, $M_{j+1} > M_j > 0$, $j \in \mathbb{N}$, $\lim_{j \rightarrow \infty} M_j = \infty$, such that $X \setminus K_j$ is connected for all $j \in \mathbb{N}$.*

- (a) *If $\dim X \geq 3$ then there is a proper holomorphic embedding $\mathcal{R} \hookrightarrow X$*
- (b) *If $\dim X = 2$ then there is a proper holomorphic immersion $\mathcal{R} \rightarrow X$.*

We can also avoid the assumption that $X \setminus K_j$ is connected if \mathcal{R} admits a subharmonic exhaustion function with finitely many critical points.

Our main motivation is the recent work of the first author [5], who has shown that Stein manifolds are characterized by their endomorphism semigroups as long as they admit a proper holomorphic embedding of the complex line \mathbb{C} . As a corollary to our main theorem we thereby obtain:

Theorem (5.6). *Let X and Y be complex manifolds and $\Phi : \text{End}(X) \rightarrow \text{End}(Y)$ an epimorphism of semigroups of holomorphic endomorphisms. If X is a Stein manifold with density or volume density property and of dimension at least 3, then there exists a unique $\varphi : X \rightarrow Y$ which is either biholomorphic or antibiholomorphic and such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$.*

Secondly, our work generalizes recent work of Drinovec-Drnovšek and Forstnerič [8] who have proven that *bordered* Riemann Surfaces immerse properly into Stein manifolds. Note that, due to hyperbolicity, the complex plane does not embed in all Stein manifolds, so some extra structure (e.g. the density property) is needed. One might ask whether our main theorem holds with X an Oka manifold instead.

Thirdly, it was conjectured by Schoen and Yao [18] that no proper harmonic map could exist from the unit disk onto \mathbb{R}^2 . The conjecture was recently disproved by Alarcón and Galvéz [1], but a much stronger result follows easily from our main theorem:

Theorem (5.7). *Let \mathcal{R} be any open Riemann surface. Then \mathcal{R} admits a proper harmonic mapping into \mathbb{R}^2 .*

Finally, it is of general interest to find new methods to produce proper holomorphic maps from Riemann surfaces into complex manifolds, due to the long standing open problem whether any open Riemann surface admits a proper holomorphic embedding in \mathbb{C}^2 .

2. DENSITY PROPERTY

The density property was introduced in Complex Geometry by Varolin [21], [22]. For a survey about the current state of research related to density property and Andersén–Lempert theory, we refer to Kaliman and Kutzschebauch [16].

Definition 2.1. A complex manifold X has the *density property* if in the compact-open topology the Lie algebra $\text{Lie}_{\text{hol}}(X)$ generated by completely integrable holomorphic vector fields on X is dense in the Lie algebra $VF_{\text{hol}}(X)$ of all holomorphic vector fields on X .

Definition 2.2. Let a complex manifold X be equipped with a holomorphic volume form ω (i.e. ω is nowhere vanishing section of the canonical bundle). We say that X has the *volume density property* with respect to ω if in the compact-open topology the Lie algebra $\text{Lie}_{\text{hol}}^{\omega}(X)$ generated by completely integrable holomorphic vector fields ν such that $\nu(\omega) = 0$, is dense in the Lie algebra $VF_{\text{hol}}^{\omega}(X)$ of all holomorphic vector fields that annihilate ω .

The following theorem is the central result of Andersén–Lempert theory (originating from works of Andersén and Lempert [3], [4]), and is given in the following form in [16] by Kaliman and Kutzschebauch, but essentially (for \mathbb{C}^n) proven already in [13] by Forstnerič and Rosay.

Theorem 2.3. *Let X be a Stein manifold with the density (resp. volume density) property and let Ω be an open subset of X . In case of volume density property further assume that $H^{n-1}(\Omega, \mathbb{C}) = 0$. Suppose that $\Phi : [0, 1] \times \Omega \rightarrow X$ is a \mathcal{C}^1 -smooth map such that*

- (1) $\Phi_t : \Omega \rightarrow X$ is holomorphic and injective (and resp. volume preserving) for every $t \in [0, 1]$
- (2) $\Phi_0 : \Omega \rightarrow X$ is the natural embedding of Ω into X
- (3) $\Phi_t(\Omega)$ is a Runge subset of X for every $t \in [0, 1]$

Then for each $\varepsilon > 0$ and every compact subset $K \subset \Omega$ there is a continuous family $\alpha : [0, 1] \rightarrow \text{Aut}(X)$ of holomorphic (and resp. volume preserving) automorphisms of X such that

$$\alpha_0 = \text{id} \text{ and } |\alpha_t - \Phi_t|_K < \varepsilon$$

for every $t \in [0, 1]$.

Remark 2.4. In the case of the volume density property it is enough to assume that $H^{n-1}(\Omega', \mathbb{C}) = 0$ for all connected components Ω' of Ω where Φ is not the identity map. The assumption is used to solve a certain differential equation which is trivially solvable on the components where Φ is the identity.

Among one of the many results following from this theorem, we only need the following, first given by Varolin [22]:

Proposition 2.5. *Let X be a Stein manifold of dimension $n \geq 2$ with density (resp. volume density) property, K be a compact in X , and $x, y \in X$ be two points outside the convex hull of K . Suppose that $x_1, \dots, x_m \in K$. Then there exists a (resp. volume-preserving) holomorphic automorphism Ψ of X such that $\Psi(x_i) = x_i$ for every $i = 1, \dots, m$, $\Psi|_K : K \rightarrow X$ is as close to the natural embedding as we wish, and $\Psi(y) = x$.*

Corollary 2.6. *Let X be a Stein manifold of dimension $n \geq 2$ with density (resp. volume density) property, then its group of holomorphic automorphisms acts m -transitively for any $m \in \mathbb{N}$.*

Stein manifolds with (volume) density property are *elliptic* in the sense of Gromov and satisfy the so-called Oka–Grauert–Gromov principle.

Definition 2.7. Let X be a complex manifold. It is said to have the *Convex Approximation Property*, if every holomorphic map of a compact convex set $K \subset \mathbb{C}^n$ in X can be approximated uniformly on K by entire holomorphic maps $\mathbb{C}^n \rightarrow X$.

In more recent terminology introduced by Forstnerič [15], a manifold satisfying the Convex Approximation Property is called an *Oka manifold*. All elliptic Stein manifolds, in particular those with (volume) density property, are Oka manifolds.

Oka manifolds satisfy the following (see Drinovec-Drnovšek and Forstnerič [10]):

Theorem 2.8. *Let S be a Stein manifold and $D \subset\subset S$ a strongly pseudoconvex domain with C^ℓ boundary ($\ell \geq 2$) whose closure \overline{D} is $\mathcal{O}(S)$ -convex, and let Y be an Oka manifold. Let $r \in \{0, 1, \dots, \ell\}$ and let $f : S \rightarrow Y$ be a C^r -map which is holomorphic in D . Then f can be approximated in the $C^r(\overline{D}, Y)$ -topology by holomorphic maps $S \rightarrow Y$ which are homotopic to f .*

3. APPROXIMATING IN THE NON-CRITICAL CASE

The goal of this section is to prove the following approximation result.

Proposition 3.1. *Let $\mathcal{R}_0 \subset\subset \mathcal{R}_1 \subset\subset \mathcal{R}_2$ be bordered Riemann surfaces, let X be a Stein manifold with the density or volume density property, and let $K \subset X$ be a holomorphically convex compact set. Let $f : \overline{\mathcal{R}_1} \rightarrow X$ be a holomorphic immersion, and assume that $f(\overline{\mathcal{R}_1} \setminus \overline{\mathcal{R}_0}) \subset X \setminus K$. Let Γ be one of the boundary components of \mathcal{R}_1 , and $\mathcal{A} \subset \mathcal{R}_2 \setminus \overline{\mathcal{R}_0}$ be an annulus containing Γ .*

Then for any compact subset L of \mathcal{A} and any $\varepsilon > 0$ there exists a holomorphic immersion $g : \mathcal{R}_1 \cup L \rightarrow X$ such that the following holds:

- (1) $\|g - f\|_{\mathcal{R}_0} < \varepsilon$, and
- (2) $g((\mathcal{R}_1 \cup L) \setminus \overline{\mathcal{R}_0}) \subset X \setminus K$.

This will in turn depend on the following result:

Proposition 3.2. *Let $\mathcal{R}_0 \subset \subset \mathcal{R}_1 \subset \subset \mathcal{R}_2$ be bordered Riemann surfaces, let X be a Stein manifold with the density or volume density property, and let $K \subset X$ be a holomorphically convex compact set. Let $f : \overline{\mathcal{R}_1} \rightarrow X$ be a holomorphic immersion, and assume that $f(\overline{\mathcal{R}_1 \setminus \mathcal{R}_0}) \subset X \setminus K$. Let Γ be one of the boundary components of \mathcal{R}_1 , let $V \subset \mathcal{R}_2 \setminus \mathcal{R}_0$ be an open set containing Γ , and assume that $f|_V$ is an embedding, with $f(\overline{V})$ not intersecting $K \cup f(\overline{\mathcal{R}_0})$.*

Let $U \subset \mathcal{R}_2 \setminus \mathcal{R}_0$ be a simply connected open set, $U \cap \mathcal{R}_1 \subset V \cap \mathcal{R}_1$, $U \cap \mathcal{R}_1$ connected, and assume that we are given a C^∞ -smooth isotopy $\varphi(\cdot, t) : U \rightarrow \mathcal{R}_2 \setminus \mathcal{R}_0$, $t \in [0, 1]$, of injective holomorphic maps, such that the following holds:

- (1) $\varphi(\cdot, 0) = \text{id}_U$,
- (2) $\varphi(U, 1) \subseteq V$,
- (3) $\varphi(U \setminus \mathcal{R}_1, t) \subseteq \mathcal{R}_2 \setminus \mathcal{R}_1$ for all $t \in [0, 1]$, and
- (4) $\varphi(U \cap \mathcal{R}_1, t) \subseteq V \cap \mathcal{R}_1$ for all $t \in [0, 1]$.

Then for any compact subset L of U and any $\varepsilon > 0$ there exists a holomorphic immersion $g : \mathcal{R}_1 \cup L \rightarrow X$ such that the following holds:

- (1) $\|g - f\|_{\mathcal{R}_0} < \varepsilon$, and
- (2) $g((\mathcal{R}_1 \cup L) \setminus \mathcal{R}_0) \subset X \setminus K$.

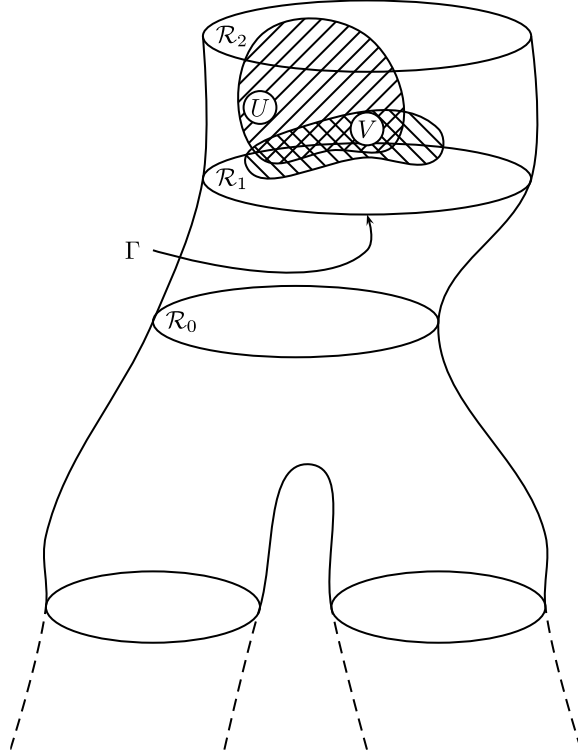


FIGURE 1.

We cite Theorem 4.1 from Forstnerič [14] which will be needed in the proof:

Theorem 3.3. *Let A and B be compact sets in a complex manifold X such that $D = A \cup B$ has a basis of Stein neighbourhoods in X and $\overline{A \setminus B \cap B \setminus A} = \emptyset$. Given an open set $\tilde{C} \subseteq X$ containing $C := A \cap B$ there exist open sets $A' \supseteq A, B' \supseteq B, C' \supseteq C$ with $C' \subseteq A' \cap B' \subseteq \tilde{C}$, satisfying the following: For every injective*

holomorphic map $\gamma : \tilde{C} \rightarrow X$ which is sufficiently uniformly close to the identity on \tilde{C} there exist injective holomorphic maps $\alpha : A' \rightarrow X, \beta : B' \rightarrow X$, uniformly close to the identity on their respective domains and satisfying

$$\gamma = \beta \circ \alpha^{-1}$$

In analogy to the classical splitting for Cartan pairs, we call such A and B satisfying the assertions of the theorem also a *Cartan pair*.

We will also need the following:

Proposition 3.4 (Hilfssatz 11 in [12]). *Let X be a Stein manifold and Y be an analytic submanifold. Then there exists a biholomorphic map of an open neighbourhood of Y in X onto an open neighbourhood of the zero section of the normal bundle of Y in X , mapping Y biholomorphically to the zero section.*

Corollary 3.5. *Let X be a complex manifold of dimension n and let $f : \overline{\mathcal{R}} \rightarrow X$ be an immersion of a bordered Riemann surface \mathcal{R} . Then there exists a holomorphic immersion $F : \overline{\mathcal{R}} \times \mathbb{D}^{n-1} \rightarrow X$ such that $F|_{\mathcal{R} \times \{0\}} = f$.*

Proof. Let \tilde{X} be a complex manifold with an embedding $\tilde{f} : \overline{\mathcal{R}} \rightarrow \tilde{X}$ and an immersion $\rho : \tilde{X} \rightarrow X$ such that $f = \rho \circ \tilde{f}$. By Siu's theorem [20] we may assume that \tilde{X} is Stein, and so the proposition applies to give an embedding \tilde{F} into \tilde{X} . Note that any vector bundle over an open Riemann surface is trivial, and define $F = \rho \circ \tilde{F}$. \square

Lemma 3.6. *Let X be a Stein manifold, let $K \subset X$ be a holomorphically convex compact set, and let $f : \overline{\mathbb{D}} \rightarrow X \setminus K$ be an embedding. Let $V' \subset X \setminus K$ be an open neighbourhood of $f(\overline{\mathbb{D}})$ and assume given an isotopy of holomorphic injections $\phi_t : V' \rightarrow X \setminus K$. Then there exists an open neighbourhood $V'' \subseteq V'$ of $f(\overline{\mathbb{D}})$ and an open neighbourhood W of K such that $\Omega_t = \phi_t(V'' \cup W)$ is a Runge domain in X for all $t \in [0, 1]$.*

Proof. It is a well known fact that $K \cup \phi_t(f(\overline{\mathbb{D}}))$ is holomorphically convex for each fixed $t \in [0, 1]$ (for lack of a reference we include an argument below). The result is then a consequence of Lemma 2.2 in [13] formulated for X instead of \mathbb{C}^n (the proof in the case of a Stein manifold is identical).

We now show that $K \cup \phi_t(f(\overline{\mathbb{D}}))$ is holomorphically convex for each fixed $t \in [0, 1]$. Let $r > 1$ be chosen close enough to 1 such that $(\phi_t \circ f) : \overline{\mathbb{D}}_r \rightarrow X \setminus K$ is an embedding, let $\Sigma = \phi_t(f(b\mathbb{D}))$, and $\Sigma' = \phi_t(f(b\mathbb{D}_r))$. We want to show that $\widehat{K \cup \Sigma} = K \cup \phi_t(f(\overline{\mathbb{D}}))$.

By Theorem 12.5. in [2] and the fact that X embeds properly in \mathbb{C}^N for N sufficiently large, we have that $\widehat{K \cup \Sigma'} \setminus (K \cup \Sigma')$ (resp. $\widehat{K \cup \Sigma} \setminus (K \cup \Sigma)$) is a one-dimensional analytic subset of $X \setminus (K \cup \Sigma')$ (resp. $X \setminus (K \cup \Sigma)$). Note first that $\widehat{K \cup \Sigma}$ cannot contain a relatively open subset of $\phi_t(f(\mathbb{D}_r \setminus \overline{\mathbb{D}}))$. If it did, it would, by the identity principle for analytic sets, contain $\phi_t(f(\mathbb{D}_r \setminus \overline{\mathbb{D}}))$, and so $\widehat{K \cup \Sigma} \setminus K$ would be an analytic subset of $X \setminus K$. This is impossible since K is holomorphically convex. Since $K \cup \phi_t(f(\overline{\mathbb{D}})) \subset \widehat{K \cup \Sigma'}$ we get that

$$\widehat{K \cup \phi_t(f(\overline{\mathbb{D}})) \setminus (K \cup \phi_t(f(\overline{\mathbb{D}})))} \cap (K \cup \phi_t(f(\overline{\mathbb{D}}))) = K \cup A,$$

where A is a finite set of points. By Rossi's local maximum principle we have that $\widehat{K \cup \phi_t(f(\overline{\mathbb{D}}))} = (K \cup \phi_t(f(\overline{\mathbb{D}}))) \cup \widehat{K \cup A}$ which implies that $K \cup \phi_t(f(\overline{\mathbb{D}}))$ is holomorphically convex. \square

Proof of Proposition 3.2.

Since X is an Oka manifold we may assume, by approximation, that f is already defined on $\overline{\mathcal{R}_2}$; the task is to find an approximation which achieves (2). Note that $K \cup f(\overline{\mathcal{R}_0})$ is holomorphically convex.

Define $A := \overline{\mathcal{R}_1}$, and let $B \subset \mathcal{R}_2$ be a Stein compact such that the pair A, B is a Cartan pair as in Theorem 3.3, $A \cap B$ simply connected and contained in V , and $L \subset (A \cup B)^\circ$. We will approximate f on a certain thickening of $A \cup B$ in $\mathcal{R}_2 \times \mathbb{C}^{n-1}$ which will allow us to exploit the density property of X .

Since f is an immersion we have by Corollary 3.5 that f extends to an immersion

$$F : \mathcal{R}_2 \times \mu \cdot \mathbb{D}^{n-1} \rightarrow X,$$

such that $F|_{\mathcal{R}_2 \times \{0\}} = f$. We may assume that $F|_{\overline{V} \times \mu \cdot \mathbb{D}^{n-1}}$ is an embedding whose image does not intersect $K \cup f(\overline{\mathcal{R}_0})$.

Set $\tilde{\omega} := F^* \omega$. By choosing μ_1 small enough we have that φ_t extends to an isotopy $\phi_t : U \times \mu_1 \cdot \mathbb{D}^{n-1} \rightarrow (\mathcal{R}_2 \setminus \mathcal{R}_0) \times \mu \cdot \mathbb{D}^{n-1}$ of the form

$$\phi_t(x, w) = (\varphi_t(x), \sigma_t(x, w)), \sigma_t(x, 0) = 0,$$

and such that $\phi_t^* \tilde{\omega} = \tilde{\omega}$ for all $t \in [0, 1]$. For $\mu_1 > 0$ small enough σ_t can be found easily in such local coordinates where ω is the standard volume form.

Now the following is our strategy: note that there is an open neighbourhood W of $C = A \cap B$, relatively compact in V , such that on the image $\Omega := F(W \times \mu_1 \cdot \mathbb{D}^{n-1})$ we have a well defined isotopy

$$\Phi_t := F \circ \phi_t \circ F^{-1} : \Omega \rightarrow X \setminus (K \cup f(\overline{\mathcal{R}_0})).$$

Choose W such that $H^{n-1}(W \times \mu_1 \cdot \mathbb{D}^{n-1}, \mathbb{C}) = 0$, and note that $\Phi_t^* \omega = \omega$ for all t . Note also that the composition $F_B := F \circ \phi_1$ is well defined near $B \times \mu_1 \cdot \mathbb{D}^{n-1}$. We will approximate Φ_1 well enough by an automorphism Λ of X , essentially fixing $K \cup f(\overline{\mathcal{R}_0})$, such that the map $\Lambda \circ F$ may be glued with minor perturbations to the map F_B .

By Lemma 3.6 there exists a neighbourhood Ω_1 of $K \cup f(\overline{\mathcal{R}_0})$ and a neighbourhood $\Omega_2 \subset \Omega$ of $f(\overline{W})$ such that $\Phi_t(\Omega_1 \cup \Omega_2)$ is Runge for each t , where $\Phi_t|_{\Omega_1} \equiv \text{id}$. Fix a $0 < \mu_2 < \mu_1$ such that $F(\overline{W} \times \mu_2 \cdot \mathbb{D}^{n-1}) \subset \Omega_2$. For any $0 < \delta < \mu_2$ let A_δ, B_δ denote the Cartan pairs $A \times \delta \cdot \mathbb{D}^{n-1}$ and $B \times \delta \cdot \mathbb{D}^{n-1}$ respectively. Let \tilde{C}_δ be a neighbourhood of $C_\delta := A_\delta \cap B_\delta$ contained in $W \times \mu_2 \cdot \mathbb{D}^{n-1}$, and let C'_δ be the corresponding neighbourhood of C_δ in Theorem 3.3.

Now let Λ_j be a sequence of automorphisms of X converging uniformly to Φ_1 near $F(C'_\delta)$ and such that each Λ_j stays uniformly close to the identity near $K \cup f(\overline{\mathcal{R}_0})$. This is possible by the (volume) density property (Theorem 2.3 and the remark following it) of X and the choices made above.

Then $\gamma_j := F^{-1} \circ \Lambda_j^{-1} \circ F \circ \phi_1$ converges to the identity uniformly on C'_δ . Decompose $\gamma_j = \alpha_j \circ \beta_j^{-1}$ using Theorem 3.3. Define $G_j := \Lambda_j \circ F \circ \alpha_j$ on A'_δ and $G_j := F \circ \phi_1 \circ \beta_j$ on B'_δ . Now put $G := G_j$ for a large enough j and then $g := G|_{(A \cup B) \times \{0\}}$. \square

Proof of Proposition 3.1.

- (1) The extension from f to g will be achieved in two steps, attaching in each step a simply connected domain in \mathcal{R}_2 using Proposition 3.2. Since f is defined on $\overline{\mathcal{R}_1} \subset \mathcal{R}_2$, it extends immersively to a neighbourhood $\mathcal{R}'_1 \subset \subset \mathcal{R}_2$ such that f is injective on a neighbourhood $V \subset \mathcal{R}_2$ of $b\mathcal{R}'_1$ which is generically the case.

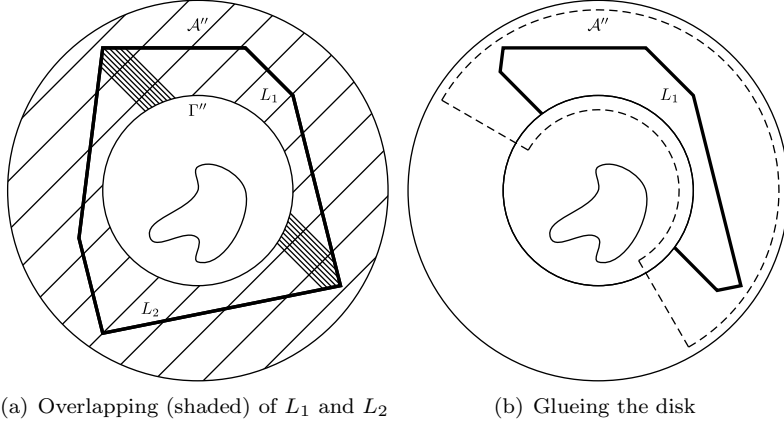


FIGURE 2. Glueing an annulus to a bordered Riemann surface

- (2) We embed the annulus \mathcal{A} in \mathbb{C} as a planar domain and arrange it by an uniformizing map such that $\mathcal{A} \setminus \mathcal{R}'_1 \rightarrow \mathcal{A}'$. Now we can work entirely in \mathbb{C} and identify all subsets of \mathcal{A} with subsets of \mathbb{C} in order to give the sets U and isotopies φ needed for Proposition 3.2. The curve Γ is mapped to a smooth curve Γ' in the image \mathcal{A}' of \mathcal{A} . Let D denote the bounded domain in \mathbb{C} bordered by Γ' . Then D is homeomorphic to a disk and $\mathcal{A}' \setminus D$ is again an annular region, which we can identify with an annulus $\mathcal{A}'' =: \ell\mathbb{D} \setminus \overline{\mathbb{D}}$, $\ell > 0$ via another uniformizing map. This map is a \mathcal{C}^∞ -smooth diffeomorphism up to the boundary and therefore extends to a small neighbourhood. The set $L \cap \mathcal{A}''$ can be written as a union of two compact overlapping sets L_1 and L_2 which are both homotopic to disks inside in \mathcal{A}'' , as depicted in figure 2(a).

Set $f_0 := f$. The immersion after the first extension to any compact $L_1 \subset U_1 \subset \mathcal{A}'$ will be denoted by f_1 , and after the second extension to $L_2 \subset U_2 \subset \mathcal{A}'$ by $f_2 = g$.

- (3) Define U_1 to be

$$U_1 := \{r \cdot e^{i\theta} \in \mathbb{C} : (1 - \delta) < r < \ell(1 - \delta), \alpha < \theta < \beta\}$$

where $1 > \delta > 0$, and $\alpha, \beta \in \mathbb{R}$ are such that $L \subset (1 - \delta)\ell\mathbb{D}$. The \mathcal{C}^∞ -smooth isotopy $\varphi_1(z, t)$, $t \in [0, 1]$, of holomorphic injections is given explicitly in equation (1) below:

$$\varphi_1(z, t) = \exp(\log(z) \cdot ((\gamma - 1) \cdot t + 1)) \quad (1)$$

with \log defined on $\mathbb{C} \setminus \mathbb{R}^-$ and suitable angle $\gamma \in \mathbb{R}$. We apply Proposition 3.2 to extend $f_0 : \mathcal{R}'_1 \subset \subset \mathcal{R}_2 \rightarrow X$ approximately up to $\varepsilon/2$ to $L_1 \subset U_1$ using the isotopy φ_1 and denote the approximation by f_1 .

- (4) Now consider $\mathcal{A}'' \setminus L_1$ which is again homeomorphic to an annulus and can be mapped to $\mathcal{A}''' =: \ell'\mathbb{D} \setminus \overline{\mathbb{D}}$, $\ell' > 0$ by another uniformizing map. Then we are back in the previous situation and define U_2 and φ_2 the same way. This leads to the desired approximation $g = f_2$. \square

4. APPROXIMATING IN THE CRITICAL CASE

Definition 4.1. A strongly subharmonic exhaustion function ρ of a Riemann surface \mathcal{R} is said to be a Morse function with *nice singularities* if any critical point ξ

is either a local minimum, or there exist local coordinates $z = x + iy : U_\xi \rightarrow \mathbb{C}$ such that ρ is of the form

$$\rho(z) = \rho(\xi) + x^2 - \mu \cdot y^2,$$

for some $\mu \in (0, 1]$.

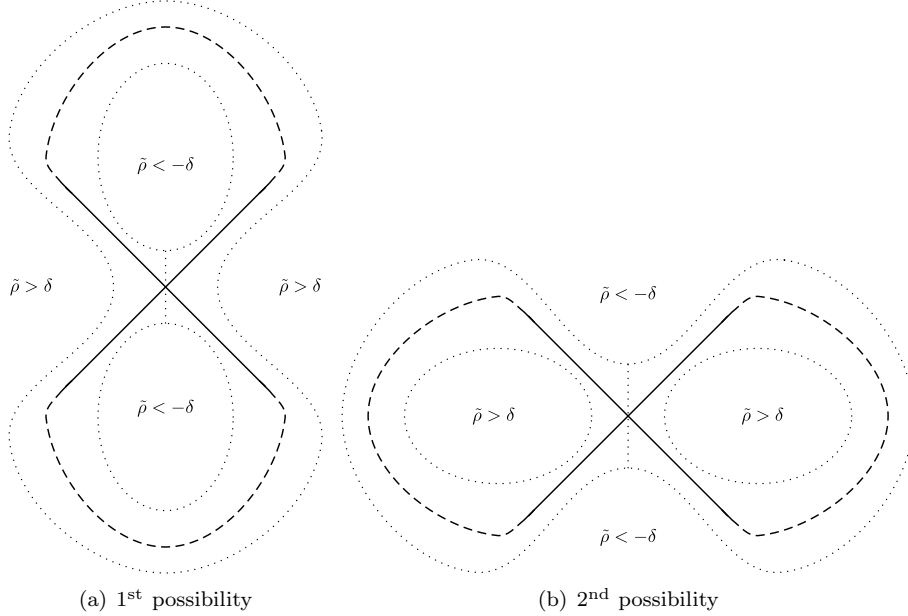


FIGURE 3. Critical points for the strongly plurisubharmonic exhaustion function $\rho : \mathcal{R} \rightarrow \mathbb{R}$ at $\rho = 0$ and the level sets of ρ

Proposition 4.2. *Let \mathcal{R} be an open connected Riemann surface, and let $\rho \in C^\infty(\mathcal{R})$ be a Morse exhaustion function with nice singularities. Let $\xi \in \mathcal{R}$ be a critical point of ρ which is not a local minimum, and let $c = \rho(\xi)$. Then there exists a $\delta > 0$ such that the following holds: Let X be a Stein manifold, let $K \subset X$ be a holomorphically convex compact set with $X \setminus K$ connected, let $f : \mathcal{R}_{c-\delta} \rightarrow X$ be a holomorphic immersion with $f(b\mathcal{R}_{c-\delta}) \in X \setminus K$, and let $\epsilon > 0$. Then there exists a holomorphic immersion $\tilde{f} : \mathcal{R}_{c+\delta} \rightarrow X$ such that*

- (1) $\|\tilde{f} - f\|_{\mathcal{R}_{c-\delta}} < \epsilon$
- (2) $\tilde{f}(\mathcal{R}_{c+\delta} \setminus \mathcal{R}_{c-\delta}) \subset X \setminus K$.

Proof. We will describe first how to cross the connected component of $\{\rho = c\}$ which contains the critical point ξ ; crossing the other components are done by applying Proposition 3.1. Let $z : U_\xi \rightarrow \mathbb{C}$ be local coordinates with $z(\xi) = 0$, and such that, in local coordinates, $\rho(x, y) = x^2 - \mu \cdot y^2$, $0 < \mu \leq 1$ (for simplicity we assume $\rho(\xi) = 0$). To get a clear picture of what is going on we start by embedding a neighbourhood of the connected component Γ of $\{\rho = 0\}$ that intersects U_ξ into \mathbb{C} . In local coordinates $\Gamma_\xi = \Gamma \cap U_\xi$ is the union of the straight lines $\gamma_\pm = \{x = \pm \sqrt{\mu} \cdot y\}$. Let $\tilde{\gamma}_\pm$ denote the preimages of these lines in U_ξ . Since $\{\rho = 0\}$ is smooth outside ξ we have that Γ is obtained by attaching two smooth arcs $l_j, j = 1, 2$, to the endpoints of the arcs $\tilde{\gamma}_\pm$. This means that Γ may be written as the union of two closed curves Γ_1 and Γ_2 intersecting at a single point ξ . Assume that Γ_1 is obtained by attaching an end-point of l_1 to the end-point of $\tilde{\gamma}_+$ which in local coordinates

lies in the upper half plane H^+ . Then the other end-point of l_1 must be attached to one of the end-points of $\tilde{\gamma}_-$. Otherwise, since \mathcal{R} may be oriented, there would exist a path starting in $\{\rho < 0\}$, never crossing $\{\rho = 0\}$, and ending up in $\{\rho > 0\}$. This means that the coordinate function z may be extended smoothly to $\Gamma_1 \setminus U_\xi$ such that the image $z(\Gamma_1 \setminus \xi)$ is either completely contained in H^+ or in the right half-plane H^R . Likewise, the coordinate function z may be extended smoothly to $\Gamma_2 \setminus U_\xi$ such that the image $z(\Gamma_2 \setminus \xi)$ is either completely contained in H^- or in H^L . Approximating the extended map z using Mergelyan's Theorem, we get an embedding $\tilde{z} : \Omega \rightarrow \mathbb{C}$ of an open neighbourhood Ω of Γ , such that the image together with the level sets of the strictly subharmonic function $\tilde{\rho} = \rho \circ \tilde{z}^{-1}$ is described by Figure 3.

Let $\tilde{\Omega} = \tilde{z}(\Omega)$, $\tilde{\Gamma} = \tilde{z}(\Gamma)$. By Figure 3 it is clear that in both cases we may assume that $\tilde{\Omega}$ (topologically) is a disk with two holes taken out.

We now consider case (a). Choosing $\delta > 0$ small enough it is clear that $\{\tilde{\rho} = -\delta\}$ is the union of two smooth closed curves, one in each bounded component of $\mathbb{C} \setminus \tilde{\Gamma}$, and $\{\tilde{\rho} = \delta\}$ is a single smooth closed curve in the unbounded component of $\mathbb{C} \setminus \tilde{\Gamma}$. Fix a δ small enough, and let $\tilde{\sigma}$ denote the vertical straight line segment passing through the origin and connecting the two components of $\{\tilde{\rho} < -\delta\}$. Define $\sigma = \tilde{z}^{-1}(\tilde{\sigma})$. It is clear from Figure 3, that $\{\tilde{\rho} < \delta\} \setminus ((\tilde{\rho} \leq -\delta) \cup \tilde{\sigma})$ has the topological type of an annulus. Therefore, by Proposition 3.1, it is enough to approximate f by a map g which is holomorphic on a neighbourhood of $C = \{\rho \leq -\delta\} \cup \sigma$, with $g(\sigma) \cap K = \emptyset$.

Note first that C is holomorphically convex in \mathcal{R} : if $\zeta \in \mathcal{R} \setminus C$ we want to find an continuous path in $\mathcal{R} \setminus C$ between ζ and a point in $\{\rho > 0\}$. Clearly there is a path in $\mathcal{R} \setminus \{\rho \leq -\delta\}$ connecting ζ and some point in $\{\rho > 0\}$. If this path does not intersect σ we are done. If it intersects σ , it is clear from Figure 3 that we may modify the path so that it does not.

Now since $X \setminus K$ is connected we may find a smooth extension \tilde{f} of f to σ such that $\tilde{f} = f$ on some neighbourhood of $\{\rho \leq -\delta\}$ and such that $\tilde{f}(\sigma) \in X \setminus K$. By Mergelyan's Theorem and the fact that X is Stein, we have that \tilde{f} may be approximated arbitrarily well by holomorphic maps.

The case (b) is now dealt with in a similar manner, note only that in this case, $\{\tilde{\rho} < \delta\} \setminus C$ is the union of two disjoint annuli, hence Proposition 3.1 must be applied twice after extending (approximating) the map to σ .

Finally we also need to approximate when crossing the remaining components of $\{\rho = 0\}$. If δ is chosen small enough we have that there are no critical points other than ξ in $\{-\delta < \rho < \delta\}$, and so this approximation is furnished by Proposition 3.1. \square

5. MAIN THEOREM AND APPLICATIONS

Theorem 5.1. *Let X be a Stein manifold with density property or with volume density property and \mathcal{R} an open Riemann surface. If $\mathcal{R} \not\cong \mathbb{C}$, then assume further there exists a strongly plurisubharmonic exhaustion function τ of X with increasing compact sublevel sets $K_j = \tau^{-1}([-\infty, M_j])$, $M_{j+1} > M_j > 0$, $j \in \mathbb{N}$, $\lim_{j \rightarrow \infty} M_j = \infty$, such that $X \setminus K_j$ is connected for all $j \in \mathbb{N}$.*

- (a) *If $\dim X \geq 3$ then there is a proper holomorphic embedding $\mathcal{R} \hookrightarrow X$.*
- (b) *If $\dim X = 2$ then there is a proper holomorphic immersion $\mathcal{R} \rightarrow X$.*

The main ingredients for the proof are the Propositions 3.1 and 4.2 from the previous sections. They will be used in an inductive framework provided by Lemma 6.3 from Drinovec-Drnovšek and Forstnerič [8] which we cite here:

Lemma 5.2. *Let X be an irreducible complex space of dimension $n \geq 2$, and let $\tau : X \rightarrow \mathbb{R}$ be a smooth exhaustion function which is $(n-1)$ -convex on $\{x \in X : \tau(x) > M_1\}$. Let \mathcal{R} be a finite Riemann surface, let P be an open set in \mathbb{C}^N containing 0, and let $M_2 > M_1$. Assume that $f : \overline{\mathcal{R}} \times P \rightarrow X$ is a spray of maps of class $\mathcal{A}^2(\mathcal{R})$ with the exceptional set $\sigma \subset \mathcal{R}$ of order $k \in \mathbb{N}$, and $U \subset \mathcal{R}$ is an open subset such that $f_0(z) \in \{x \in X_{\text{reg}} : \tau(x) \in (M_1, M_2)\}$ for all $z \in \overline{\mathcal{R}} \setminus U$. Given $\varepsilon > 0$ and a number $M_3 > M_2$, there exist a domain $P' \subset P$ containing $0 \in \mathbb{C}^N$ and a spray of maps $g : \overline{\mathcal{R}} \times P' \rightarrow X$ of class $\mathcal{A}^2(\mathcal{R})$, with exceptional set σ of order k , satisfying the following properties:*

- (1) $g_0(z) \in \{x \in X_{\text{reg}} : \tau(x) \in (M_2, M_3)\}$ for $z \in b\mathcal{R}$,
- (2) $g_0(z) \in \{x \in X : \tau(x) > M_1\}$ for $z \in \overline{\mathcal{R}} \setminus U$,
- (3) $d(g_0(z), f_0(z)) < \varepsilon$ for $z \in U$, and
- (4) f_0 and g_0 have the same k -jets at each of the points in σ .

Moreover, g_0 can be chosen homotopic to f_0 .

First we note, that X in our case will be a Stein manifold and therefore τ can be taken to be a strongly plurisubharmonic exhaustion function, and we have $X = X_{\text{reg}}$ as well. The existence of a metric follows in the general case from paracompactness, but in our case of Stein manifolds we can work with the restriction of an euclidean norm. We also cite from [9] their definition of spray of maps of class $\mathcal{A}^2(\mathcal{R})$:

Definition 5.3. Assume that X is a complex manifold, \mathcal{R} is a relatively compact strongly pseudoconvex domain with \mathcal{C}^2 boundary in a Stein manifold S , and σ is a finite set of points in \mathcal{R} . A spray of maps of class $\mathcal{A}^2(\mathcal{R})$ with the exceptional set σ of order $k \in \mathbb{N}$ (and with values in X) is a map $f : \overline{\mathcal{R}} \times P \rightarrow X$, where P (the parameter set of the spray) is an open subset of a Euclidean space \mathbb{C}^m containing the origin, such that the following hold:

- (1) f is holomorphic on $\mathcal{R} \times P$ and of class \mathcal{C}^2 on $\overline{\mathcal{R}} \times P$
- (2) the maps $f(\cdot, 0)$ and $f(\cdot, t)$ agree on σ up to order k for $t \in P$, and
- (3) for every $z \in \overline{\mathcal{R}} \setminus \sigma$ and $t \in P$ the map

$$\partial_t f(z, t) : T_t \mathbb{C}^m = \mathbb{C}^m \rightarrow T_{f(z, t)} X$$

is surjective (the domination property).

The map $f_0 = f(\cdot, 0)$ is called the core (or central) map of the spray f .

In our case, S will be a fixed finite open Riemann surface and in any step \mathcal{R} the sublevel set of a strongly plurisubharmonic exhaustion function of S . The core map of the spray will be the holomorphic immersion f of $\overline{\mathcal{R}}$ into the complex manifold X of dimension n , and we construct a spray as follows: Let \mathcal{R}' be a Riemann surface with $\mathcal{R} \subset \mathcal{R}'$. Since the tangent bundle of $\overline{\mathcal{R}'}$ is trivial, we may choose a non-vanishing holomorphic vector field V on $\overline{\mathcal{R}'}$, and we let φ_t denote its flow. Define a map $\tilde{f} : \overline{\mathcal{R}} \times \mathbb{D}^{n-1} \times \delta \cdot \mathbb{D} \rightarrow \mathcal{R}' \times \mathbb{D}^{n-1}$ by $(z, t_1, t_2) \mapsto (\varphi_{t_2}(z), t_1)$. Choose an immersion $F : \overline{\mathcal{R}'} \times \mathbb{D}^{n-1} \rightarrow X$ according to Corollary 3.5, and define $f := F \circ \tilde{f}$.

Proof of Theorem 5.1.

1. Let \mathcal{R} be an open connected Riemann surface. Since \mathcal{R} is a Stein manifold we have that \mathcal{R} admits a \mathcal{C}^2 strictly subharmonic exhaustion function $\rho : \mathcal{R} \rightarrow \mathbb{R}^+$. Since strict subharmonicity is stable under small \mathcal{C}^2 -perturbations we may assume that ρ is a Morse function, meaning that all critical points of ρ are non-degenerate, and if ξ and ξ' are two critical points of ρ then $\rho(\xi) \neq \rho(\xi')$. By Lemma 2.5 in [17] we may further assume that any critical point ξ is either a local minimum, or there exist local

coordinates $z = x + iy : U_\xi \rightarrow \mathbb{C}$ such that ρ is of the form

$$\rho(z) = \rho(\xi) + x^2 - \mu \cdot y^2,$$

for some $\mu \in (0, 1)$, i.e. that ρ has only nice singularities. In the following we denote by $\{\xi_k\}_{k \in I \subseteq \mathbb{N}}$ the critical points of ρ , and by $c_k := \rho(\xi_k)$ the corresponding critical values, where I is either \mathbb{N} or a $I = [1, \dots, k_{\max}]$ for some $k_{\max} \in \mathbb{N}$. If there is only a finite number of critical points, we define inductively $c_{k+1} := c_k + 1$ for $k \geq k_{\max}$.

Let τ denote a strictly plurisubharmonic exhaustion function of the Stein manifold X . Choose a sequence of real $\varepsilon_k > 0$ such that $\sum_{k=1}^{\infty} \varepsilon_k < 1$. By

$$\mathcal{R}_\gamma := \{z \in \mathcal{R} : \rho(z) < \gamma\}, \quad \gamma \in \mathbb{R}$$

we denote the γ -sublevel set of ρ .

2. For each $k \geq 2$ we do the following: if ξ_k is a local minimum we put $\delta_k := \frac{1}{2} \min\{c_k - c_{k-1}, c_{k+1} - c_k\}$, and otherwise we choose a small δ_k according to Proposition 4.2. such that $\overline{\mathcal{R}_{c_k + \delta_k}} \setminus \mathcal{R}_{c_k - \delta_k}$ contains no other critical point of ρ than ξ_k . In the case of finitely many critical points we put $\delta_k = 0$ for $k > k_{\max}$.
3. Choose an initial embedding $f_2 : \overline{\mathcal{R}_{c_2 - \delta_2}} \rightarrow X$. This is trivial since $\mathcal{R}_{c_2 - \delta_2}$ is a disk. We will now describe an inductive procedure how to construct immersions (resp. embeddings) f_k of $\overline{\mathcal{R}_{c_k - \delta_k}}$ into X . Assume that we have constructed immersions (resp. embeddings) $f_k : \overline{\mathcal{R}_{c_k - \delta_k}} \rightarrow X$ and real numbers r_k for $k = 2, \dots, N$, $r_k \geq r_{k-1} + 1$, and assume that $f_k(\overline{\mathcal{R}_{c_k - \delta_k}} \setminus \mathcal{R}_{c_{k-1}}) \subset X \setminus K_{r_k}$, $\|f_k - f_{k-1}\|_{\overline{\mathcal{R}_{c_{k-1} - \delta_{k-1}}}} < \epsilon_{k-1}$. We will now describe the inductive step how to construct f_{N+1} .
 - (a) Choose $r_{N+1} \geq r_N + 1$ such that $X \setminus K_{r_{N+1}}$ is connected. We may assume that $f_N(b\mathcal{R}_{c_N - \delta_N}) \subset X \setminus K_{r_{N+1}}$: since f_N lives on a neighborhood of $\overline{\mathcal{R}_{c_N - \delta_N}}$, we may thicken f_N as described above, and the boundary may be pushed away using Lemma 5.2.
 - (b) In the case of finitely many critical points, if $N > k_{\max}$, we may reach the next level set by attaching finitely many annuli to \mathcal{R}_{c_N} , hence the approximation is furnished by Proposition 3.1.
 - (c) If ξ_N is a local minimum start by extending the immersion (resp. embedding) to the component of $\overline{\mathcal{R}_{c_N + \delta_N}}$ that contains ξ_N ; this is trivial because this component is a disk. Make sure that the image lies in $X \setminus K_{r_{N+1}}$. Since we may now reach $\mathcal{R}_{c_{N+1} - \delta_{N+1}}$ by attaching a finite number of annuli, the approximation f_{N+1} is furnished by Proposition 3.1. If $\dim(X) \geq 3$ the separation of points is not a problem.
 - (d) If ξ_N is not a local minimum, we may choose an initial approximation $\tilde{f}_{N+1} : \overline{\mathcal{R}_{c_N + \delta_N}} \rightarrow X$ furnished by Proposition 4.2. Now $\mathcal{R}_{c_{N+1} - \delta_{N+1}}$ may be reached by attaching a finite number of annuli, and the approximation f_{N+1} is furnished by Proposition 3.1.
 - (e) It is now clear that the limit $f := \lim_{j \rightarrow \infty} f_j$ is well defined on \mathcal{R} , and gives us the desired immersion (resp. embedding) into X . For the embedding, the sequence ϵ_j should be modified along the way to avoid self intersections in the limit.

Note that in the special case of $\mathcal{R} = \mathbb{C}$ we can choose an exhaustion function $\rho : \mathbb{C} \rightarrow \mathbb{R}_0^+$ like $\rho(z) = |z|^2$, which is obviously strongly subharmonic and has no critical points except $z = 0$. After for the initial embedding we therefore never again encounter a critical point, hence the cases (c) and (d) do never occur and we do not need to require the connectedness of $X \setminus K_j$. \square

Remark 5.4. Note that the construction would also work in the case that \mathcal{R} admits an exhaustion function ρ with finitely many critical points. We would start by embedding a totally real skeleton of \mathcal{R} , containing all critical points of ρ , into X , approximate on a small neighborhood using Mergelyan's theorem, and then proceed as above.

The main result in [5] by the first author characterizes certain Stein manifolds by their endomorphism semigroup and gives an application of our theorem using a properly embedded complex line in a Stein manifold:

Theorem 5.5. *Let X and Y be complex manifolds and $\Phi : \text{End}(X) \rightarrow \text{End}(Y)$ an isomorphism of semigroups of holomorphic endomorphisms. Then there exists a unique $\varphi : X \rightarrow Y$ which is either biholomorphic or antibiholomorphic and such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ if the following criteria are fulfilled:*

- (1) X is a Stein manifold, and
- (2) X admits a proper holomorphic embedding $i : \mathbb{C} \hookrightarrow X$.

If the automorphism group of X acts (weakly) double-transitive, it is sufficient for Φ to be an epimorphism.

From Theorem 5.1 and the preceding result and noting that a Stein manifold with (volume) density property has a double-transitive action by Proposition 2.5 resp. its Corollary 2.6, we immediately get the following result:

Theorem 5.6. *Let X and Y be complex manifolds and $\Phi : \text{End}(X) \rightarrow \text{End}(Y)$ an epimorphism of semigroups of holomorphic endomorphisms. If X is a Stein manifold with density or volume density property and of dimension at least 3, then there exists a unique $\varphi : X \rightarrow Y$ which is either biholomorphic or antibiholomorphic and such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$.*

A conjecture by Schoen and Yao [18] claimed that no proper harmonic map could exist from the unit disk onto \mathbb{R}^2 . The conjecture was recently disproven by Alarcón and Galvéz [1], but a much stronger result follows easily from our main theorem:

Theorem 5.7. *Let \mathcal{R} be any open Riemann surface. Then \mathcal{R} admits a proper harmonic mapping into \mathbb{R}^2 .*

Proof. The Stein manifold $\mathbb{C}^* \times \mathbb{C}^*$ has the volume density property (with standard volume form $\frac{dz}{z} \wedge \frac{dw}{w}$), see [21]. According to Theorem 5.1 there exists a proper holomorphic immersion $(f_1, f_2) : \mathcal{R} \rightarrow \mathbb{C}^* \times \mathbb{C}^*$. The map $(\log |f_1|, \log |f_2|) : \mathcal{R} \rightarrow \mathbb{R}^2$ is harmonic and still proper. \square

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